JOURNAL OF COMBINATORIAL THEORY, Series A 64, 137–143 (1993)

Note

On Collineation Groups with Block Orbits

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Communicated by the Editors

Received May 6, 1991

Let G be a point-transitive collineation group of an affine plane of order n. In his book M. J. Kallaher conjectured that if $G_{\mathcal{O}}$ has a block orbit of degree at least 2 for some affine point $\mathcal{O}$ then G contains a group of translations of order n^2. In this article we prove that this conjecture is true. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let $\pi(\mathscr{P}, \mathscr{L})$ be an affine plane of order n, let G be a collineation group of π transitive on the set of affine points $\mathscr{P}$, and let $\mathcal{O} \in \mathscr{P}$. A point orbit Γ is called a block orbit of $G_{\mathcal{O}}$ if $\Gamma \cup \{\mathcal{O}\}$ is the set theoretic union of lines through $\mathcal{O}$. (See Chapter 12 of [4].) Let $g_1, ..., g_r$ be the lines involved the union. Let l_∞ be the line at infinity. Then we call the integer r the degree of Γ and denote by Γ_∞ the set $\{g_i \cap l_\infty \mid 1 \leqslant i \leqslant r\}$. Clearly Γ_∞ is an orbit of $G_{\mathcal{O}}$. In his book [4] M. J. Kallaher raised the following conjectures:

(A) If Γ has degree $\gamma \geqslant 2$, then π is a translation plane, and G contains the group of translations of π.

(B) If Γ has degree 1, then π is either a translation plane or a dual translation plane, and G contains the group of translations of π or the group of dual translations of π, respectively.

(C) If $G_{\mathcal{O}}$ has two block orbits, then π is a translation plane, and G contains the group of translations.

N. L. Johnson and M. J. Kallaher have shown that (A) is true if either (i) n is even, (ii) n is a nonsquare, or (iii) the degree of Γ is even

* This article was written when the author was visiting the University of Iowa during 1990–1991.

0097-3165/93 $5.00
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(Theorem 13.1 of [4]). They also have shown that (C) is true under some additional conditions. (See Theorem 13.3 of [4].) In this article we will study these conjectures and show that (A) is true without any additional condition (Section 3) but (B) and (C) are not (Section 4).

Throughout the article all sets and groups are assumed to be finite. The terminology we use is standard and can be found in [3] or [4].

2. Preliminaries

We list some known results on collineation groups which we will use later.

Result 2.1 (T. G. Ostrom and A. Wagner, Theorem 4.3 of [4]). Let π be a finite affine plane and let G be a collineation group of π. Then the following conditions are equivalent.

(a) G is transitive on the set of affine points.

(b) $G_{\mathcal{Q}}$ is transitive on the set of affine lines through $\mathcal{Q}$ for each point $\mathcal{Q} \in l_\infty$.

Result 2.2 (D. R. Hughes, Corollary 4.6.1 of [4]). Let π be a finite projective plane of order n and let G be a collineation group of π. Let l be a line of π. If $G(P, l) \neq 1$, for n of the points P on l, then $G(P, l) \neq 1$ for each point $P \in l$.

Result 2.3 (J. André, Theorem 10.1 of [4]). Let $\pi(\mathscr{P}, \mathscr{L})$ be a finite projective plane of order n, let G be a collineation group of π, and let l be a line of π. If $\Phi \equiv \{\mathcal{O} \mid \mathcal{O} \in \mathscr{P} - l, G(\mathcal{O}, l) \neq 1\} \neq \phi$, then Φ is a point orbit of $G(l, l)$ and $|G(l, l)| = |\Phi|$.

Result 2.4 [2]. Let π be a finite projective plane of order n admitting a collineation group G which fixes a flag (P, l). If G acts 2-transitively on $l - \{P\}$, then n is a power of a prime p and $G^{l-\{P\}}$ has a regular normal subgroup. Moreover the stabilizer H of a point of $l - \{P\}$ in G satisfies one of the following conditions.

(i) $H^{l-\{P\}} \leqslant \Gamma L(1, p^m)$, $n = p^m$.

(ii) $SL(2, p^m) \leqslant H^{l-\{P\}} \leqslant \Gamma L(2, p^m)$, $n = p^{2m}$.

(iii) $n \in \{2^4, 3^2, 3^4, 3^6, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\}$.

3. The Proof of the Conjecture (A)

In this section we prove the following.

Theorem. *Let π be an affine plane of order n and let G be a collineation group of π which acts transitively on the set of affine points. If $G_{\mathcal{O}}$ has a block orbit Γ and* $\deg \Gamma \geqslant 2$ *for some affine point $\mathcal{O}$, then π is a translation plane and G contains the group of translations.*

We prove the theorem by way of contradiction and a series of lemmas. Deny the theorem and fix the notations of Section 1. Then, by Theorems 12.3, 12.4, and 13.1 of [4], the following holds.

Lemma 3.1. (i) *Γ is a strong block orbit. (Γ is said to be strong if Γ_∞ is an orbit of G. See Chapter* 12 *of* [4].)

(ii) *G has no nontrivial perspectivity with center U for each $U \in \Gamma_\infty$.*

(iii) *n is odd.*

Lemma 3.2. *Let $P \in \Gamma_\infty$ and set $l = P\mathcal{O}$, $H = G_l$. Then $H \nleqslant G_{\mathcal{O}}$. Moreover $n = p^m$ for an integer m and an odd prime p and H^l (the restriction of H on l) is a 2-transitive permutation group with a regular normal subgroup.*

Proof. Set $X = G_{\mathcal{O}}$ and suppose $H \leqslant X$. Then $|G:H| = |G:G_P| \times |G_P:H| = |\Gamma_\infty| \times n$ by Result 2.1 and Lemma 3.1(i). On the other hand $|G:H| = |G:X| \times |X:H| = n^2 \times |X:X_P| = n^2 \times |\Gamma_\infty|$. Hence $n = 1$, a contradiction. Thus $H \nleqslant X$. Since X_l is transitive on $l - \{\mathcal{O}\}$, H acts 2-transitively on the set of affine points on l. Applying Result 2.4, we have the lemma.

Lemma 3.3. *H contains an involutory homology with axis l and center $\mathcal{Q} \in l_\infty - \Gamma_\infty$.*

Proof. By Corollary 3.6.1 of [4] and Lemma 3.1(iii), an involution, say τ, in the center of a Sylow 2-subgroup of H is a homology. If the axis of τ is l_∞, then $|G(l_\infty, l_\infty)| = n^2$ by the transitivity of G on the affine points of π and Result 2.3. This is not the case. Hence τ has center, say $\mathcal{Q}$, on l_∞. By Lemma 3.1(ii), $\mathcal{Q} \notin \Gamma_\infty$ and so l is the axis of τ.

Lemma 3.4. *Let $\mathcal{M}$ be the set of affine lines through $\mathcal{Q}$. Then $G_{\mathcal{Q}}$ is transitive on $\mathcal{M}$ and $H \leqslant G_{\mathcal{Q}}$.*

Proof. By Result 2.1, $G_{\mathcal{Q}}$ is transitive on $\mathcal{M}$. Assume $H \nleqslant G_{\mathcal{Q}}$. Then $G(P, l) \neq 1$ by Lemma 3.3 and Result 2.3. This is contrary to Lemma 3.1(ii).

LEMMA 3.5. $H \neq G_{\mathcal{Q}}$.

Proof. Set $R = G_{\mathcal{Q}}$ and assume $H = R$. Then $|R| = |H| = |G|/(|G : G_P| \times |G_P : H|) = |G|/|\Gamma_\infty| \times n$ by Result 2.1 and Lemma 3.1(i). On the other hand $|R| = |G|/|\mathcal{Q}^G| \geqslant |G|/(n+1-|\Gamma_\infty|)$. Hence $|G|/|\Gamma_\infty| \cdot n \geqslant |G|/(n+1-|\Gamma_\infty|)$. It follows that $|\Gamma_\infty| = 1$, a contradiction.

LEMMA 3.6. *G fixes $\mathcal{Q}$ and $G(\mathcal{Q}, \mathcal{Q}) \neq 1$.*

Proof. Set $R = G_{\mathcal{Q}}$. It follows from Lemmas 3.4 and 3.5 that H is a proper subgroup of R. Hence $G(\mathcal{Q}, \mathcal{Q}) \neq 1$ by Result 2.3 and Lemma 3.3. Applying Results 2.1 and 2.2, we have $G(\mathcal{Q}, l_\infty) \neq 1$.

Since $G_{\mathcal{O},l}$ is transitive on $l - \{\mathcal{O}\}$ and $G_{\mathcal{O},l} \leqslant H \leqslant R$, R_g is transitive on $\mathcal{M} - \{g\}$. Here $g = \mathcal{Q}\mathcal{O}$. This, together with Result 2.1, implies that R is 2-transitive on $\mathcal{M}$.

Set $\Omega = \mathcal{Q}^G$ and assume $\Omega \neq \{\mathcal{Q}\}$. Then $G(\mathcal{Q}', l_\infty) \neq 1$ for each $\mathcal{Q}' \in \Omega$. Hence $1 \neq T^{\mathcal{M}} \lhd R^{\mathcal{M}}$, where $T = G(l_\infty, l_\infty)$. The 2-transitivity of $R^{\mathcal{M}}$ forces $|T^{\mathcal{M}}| = n$. Therefore $|T| = n \times |G(\mathcal{Q}, l_\infty)| > n$. In particular $G(P, l_\infty) \neq 1$ for every $P \in \Gamma_\infty$, contrary to Lemma 3.1(ii). Thus $\Omega = \{\mathcal{Q}\}$ and hence G fixes $\mathcal{Q}$.

LEMMA 3.7. $|G(\mathcal{Q}, l_\infty)| = n$.

Proof. Set $A = G(\mathcal{Q}, \mathcal{Q})$. By Results 2.1, 2.3, and Lemmas 3.1(i), 3.6, we have $|A| = |l^G| = n \times |\Gamma_\infty| > n$. By Lemma 2.1, $|G(\mathcal{Q}, g)| = |G(\mathcal{Q}, h)|$ for all affine lines g and h through $\mathcal{Q}$. Hence $|A| = |G(\mathcal{Q}, l_\infty)| + n \times |G(\mathcal{Q}, g)|$ and so $|G(\mathcal{Q}, l_\infty)| \equiv 0 \pmod{n}$. Thus the lemma holds.

LEMMA 3.8. *$n = p^2$ and $p + 1 = 2^c$ for an integer $c \geqslant 2$.*

Proof. Let M^l be a regular normal subgroup of H^l. (See Lemma 3.2.) We may choose M so that $H \rhd M \rhd G_{(l)}$, where $G_{(l)}$ is the pointwise stabilizer of l in G. Set $C = G(\mathcal{Q}, l_\infty)$ and let B be a Sylow p-subgroup of M.

Deny the lemma. Then we can take a prime p-primitive divisor, say u, of $p^m - 1$ by Zsigmondy's result [5]. Let U be a Sylow u-subgroup of $N_H(B)$. Since $p^m - 1 (= n - 1)$ divides $|(H_{\mathcal{O}})^l|$, we have $U^l \neq 1$.

As $BC \rhd C$, $Z(BC) \cap C \neq 1$. Assume $Z(BC) \ngeqslant C$. Then $1 < |Z(BC) \cap C| < n$. Hence $[U, G(\mathcal{Q}, l_\infty)] = 1$ since $|G(\mathcal{Q}, l_\infty)| = n$ (Lemma 3.7) and u is a prime p-primitive divisor of $p^m - 1$. Therefore $U \leqslant G(P, g)$, where $g = \mathcal{O}\mathcal{Q}$. This is contrary to Lemma 3.1(ii). Thus $Z(BC) \geqslant C$. On the other hand C acts transitively on the set of affine lines through P. Thus $1 \neq B \leqslant G(P, l_\infty)$, contrary to Lemma 3.1(ii).

Proof of Theorem. By Lemma 3.8, $n = p^2$ and $p + 1 = 2^c$. Let B and C be as in the proof of Lemma 3.8. Let Y be a Sylow 2-subgroup of $N_H(B)$.

Since $H \rhd B \cdot G_{(l)}$ and $B_{(l)} = 1$, $(Y)^l$ is a Sylow 2-subgroup of H^l. In particular $|Y| \geqslant 2^{c+1}$ as $|H^l|$ is divisible by $p^2 - 1$.

If $Z(BC) \geqslant C$, we get a contradiction by a similar argument as in Lemma 3.8. Hence $|Z(BC) \cap C| = p$ and $|C/(Z(BC) \cap C)| = p$. On the other hand $4 \nmid (p-1)$. Thus we have $\mathscr{C}_Y(Z(BC) \cap C) \cap \mathscr{C}_Y(C/(Z(BC) \cap C)) \neq 1$. Here $\mathscr{C}_Y(X)$ means the centralizer of a group X in Y. Let μ be an involution in this intersection. Then $[\mu, C] = 1$ by Theorem 5.3.2 of [1]. Thus $\mu \in G(P, g)$, where $g = \mathcal{O}\mathcal{Q}$. This is contrary to Lemma 3.1(ii).

4. A Counterexample to (B) and (C)

In this section we present a counterexample to the conjectures (B) and (C). Let m be a positive integer and e an odd divisor of m. Set $K = GF(q)$, where $q = 2^m$.

A desarguesian projective plane $PG(2, q)$ of order q is a 2-dimensional projective space over K $(= GF(q))$. (See [3] for definition.) We denote a point $(u, v, w)K$ of $PG(2, q)$ by (u, v, w) and a line $\{(x, y, z)K \mid ax + by + cz = 0,\ x, y, z \in K\}$ by $[a, b, c]$. We note that a point (u, v, w) is on a line $[a, b, c]$ if and only if $au + bv + cw = 0$.

Let π $(= \pi(\mathscr{P}, \mathscr{L}))$ be an affine plane obtained from $PG(2, q)$ by deleting the line $l_\infty = [1, 0, 0]$ and the set of points on it. Then

$$\mathscr{P} = \{(1, x, y) \mid x, y \in K\}$$

$$\mathscr{L} = \{[1, x, y] \mid x, y \in K, (x, y) \neq (0, 0)\} \cup \{[0, 1, x] \mid x \in K\} \cup \{[0, 0, 1]\}.$$

Let σ be a field automorphism of K such that $x^\sigma = x^{2^d}$ for each $x \in K$. Here $d = m/e$. We define two collineation groups of π as follows: $T = \{t(x, y) \mid x, y \in K\}$, where

$$t(x, y) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & x^\sigma & 1 \end{pmatrix}$$

and $D = \{s(z) \mid z \in K^*\}$, where

$$s(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^{1+\sigma} \end{pmatrix}.$$

Remark 4.1. Set $G=TD$ and let $M\in G$. Action of M on $\mathscr{P}$ and $\mathscr{L}$ are defined as follows. (See [3].)

$$(1,x,y)^M=(1,x,y)M \quad \text{for} \quad (1,x,y)\in\mathscr{P},$$

$$[u,v,w]^M=M^{-1}\begin{pmatrix}u\\v\\w\end{pmatrix} \quad \text{for} \quad [u,v,w]\in\mathscr{L}.$$

We can easily verify the following.

(1) $s(z)^{-1}\,t(x,y)\,s(z)=t(xz,yz^{1+\sigma})$.

(2) $(1,u,v)^{t(x,y)}=(1,x+u,y+ux^\sigma+v)$.

(3) $(1,0,0)^{s(z)}=(1,0,0)$, $[0,1,0]^{s(z)}=[0,1,0]$, $[0,0,1]^{s(z)}=[0,0,1]$.

(4) The set of points on $[0,1,0]$ is $\Omega_1=\{(1,0,u)\,|\,u\in K\}$.

(5) The set of points on $[0,0,1]$ is $\Omega_2=\{(1,u,0)\,|\,u\in K\}$.

Set $P=(0,0,1)$, $\mathscr{Q}=(0,1,0)\in l_\infty$, and set $\mathscr{O}=(1,0,0)\in\mathscr{P}$.

LEMMA 4.2. (i) *G is transitive on $\mathscr{P}$ and $G_\mathscr{O}=D$.*

(ii) *Set $\Gamma=\Omega_1-\{\mathscr{O}\}$ and $\Delta=\Omega_2-\{\mathscr{O}\}$. Then Γ and Δ are block orbits of $G_\mathscr{O}$ of degree* 1.

(iii) *G fixes P but does not fix $\mathscr{Q}$.*

(iv) $|G(l_\infty,l_\infty)|=q$.

Proof. By (1), G is a group of order $q^2(q-1)$. As $G\geqslant T$, we have (i) from (2) and (3).

Since $(1,0,u)^{s(z)}=(1,0,z^{1+\sigma}\cdot u)=(1,0,z^{1+2^d}\cdot u)$ and since $(2^d+1,(2^d)^e-1)=1$, Γ is an orbit of D $(=G_\mathscr{O})$. Hence Γ is a block orbit of $G_\mathscr{O}$. Similarly, Δ is also a block orbit of $G_\mathscr{O}$.

Since T fixes P but does not fix $\mathscr{Q}$, we have (iii).

Since $D\cap G(l_\infty,l_\infty)=1$, $G(l_\infty,l_\infty)\leqslant T$. However, $T(l_\infty,l_\infty)=\{t(0,y)\,|\,y\in K\}$ and so we have (iv).

Remark 4.3. By Lemma 4.2, π is a counterexample to the conjectures (B) and (C).

Remark 4.4. The example shows that a block orbit is not always a strong block orbit. Hence this also gives a counterexample to a conjecture raised by Norman L. Johnson. (See [4, p. 117].)

Remark 4.5. The counterexample constructed above is desarguesian. The author has no non-desarguesian counterexample.

ACKNOWLEDGMENT

The author thanks Norman L. Johnson for helpful suggestions as to the preparation of this article.

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